Riemann-Cartan Geometry of Trivializable Gauge Fields

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A Riemann-Cartan structure can be associated to any SO(4) trivializable gauge field. Under certain integrability conditions, this non-Riemannian geometry may be replaced by a strictly Riemannian one. The Yang-Mills equations guarantee the existence of such a Riemannian structure. The general SO(4) trivializable solution for the SO(3) Yang-Mills equations is discussed within the geometric approach.

I. Introduction

In two preceding papers [1, 2], the general properties of SO(4) trivializable gauge fields have been investigated in great detail. Perhaps the most striking feature of these gauge fields is the emergence of a Riemannian structure, which however could be worked out only for a very special field configuration: the dimeron solution of the free Yang-Mills equations. But the Riemannian structure arises in such a natural way within the geometry framework of this specific example, that one is highly motivated to ask whether such a Riemannian geometry can be attributed to *any* trivializable gauge field configuration. Moreover, the dimeron example shows how elegantly the Yang-Mills equations may be solved by trivializable gauge fields. Hence there is some hope to obtain new solutions when the (non)-Riemannian geometry of trivializable gauge fields is better understood.

The present paper attacks these problems and presents a positive answer to the questions raised above. Especially, the emergence of a Riemannian structure is clarified in detail; it may be traced back to the foliation of the 4-space into a set of smooth 3-sheets which must carry a 3-geometry of constant curvature. The collection of the individual 3-dimensional subgeometries may then be described in terms of a 4-dimensional Riemannian structure. However this result is valid only when the characteristic distribution of the trivializable configuration is integrable. In the non-integrable case, there emerges a 4-dimensional Riemann-Cartan structure, which is the well-known Cartan generalization to a Riemannian space [3]. Both the integrable and non-integrable cases may be conveniently described in terms of fibre bundles (plane bundles).

The construction of the Riemann-Cartan geometry is organized as follows:

In Sect. II, we first show that the trivialization conditions admit the Cartan generalization of the Riemannian structure encountered by the previous dimeron geometry. This implies that the connection $\boldsymbol{\Gamma}$ for the characteristic bundle $\bar{\tau}_4$* has a non-vanishing torsion $\boldsymbol{Z}$. The latter one will in general be necessary in order that the action of the bundle curvature $\boldsymbol{R}$ of $\boldsymbol{\Gamma}$ be restricted to the characteristic distribution $\bar{\Delta}$. As a consequence of this construction, the characteristic vector field $\hat{\boldsymbol{p}}$ automatically becomes covariantly constant with respect to that connection $\boldsymbol{\Gamma}$. Further, a covariantly constant fibre metric $\bar{\boldsymbol{G}}$ can be found and extended to a metric $\boldsymbol{G}$ for the tangent bundle $\mathring{\tau}_4$ of the Euclidean base space $\boldsymbol{E}_4$, such that $\boldsymbol{\Gamma}$ becomes the Riemann-Cartan connection of $\boldsymbol{G}$. In this way, there arises a complete and consistent 4-geometry for the characteristic bundle $\bar{\tau}_4$, which itself is based upon the 3-dimensional distribution $\bar{\Delta}$.

In Sect. III, the relationship between the characteristic bundle $\bar{\tau}_4$ and the original trivializable bundle $\tilde{\tau}_4$ ("representative bundle") is discussed thoroughly. A bundle isomorphism is found by which the two "twin bundles" $\bar{\tau}_4$ and $\tilde{\tau}_4$ are revealed as being identical. As a consequence, the curvature $\boldsymbol{R}$ in $\bar{\tau}_4$ may be expressed by the corresponding fibre metric $\bar{\boldsymbol{G}}$ which yields a constant curvature geometry for $\bar{\tau}_4$. This geometry has a vanishing Weyl tensor in agreement with the analogous result for a strictly Riemannian structure. But the 4-dimensional non-Riemannian structure exhibits further "Riemannian properties" (symmetry properties of curvature tensor etc.) so that one really

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* Please notice the different symbols for the characteristic bundle $\bar{\tau}_4$ and the trivializable bundle $\tilde{\tau}_4$.

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deals with a sort of an "almost-Riemannian" structure. In any case, the 3-dimenional subgeometry on the characteristic surface (if existing) is strictly Riemannian!

In Sects. III, IV it is shown how one can introduce a four-dimensional Riemannian structure provided the characteristic distribution $\bar{\Delta}$ is integrable. The Frobenius integrability condition is satisfied automatically if the gauge field obeys the free Yang-Mills equations. Thus, the Yang-Mills equations guarantee the existence of the Riemannian structure also in four dimensions. In this case, the general shape for the Riemannian connection $\boldsymbol{\Gamma}$ can easily be found. Such a "Riemann-Yang-Mills structure" always contains a conformally flat background. The pure background solutions just coincide with the dimeron configurations.

Occasionally, the general results are exemplified with the aid of the dimeron configuration. Especially, the autoparallel and geodesic trajectories are studied for that configuration in the appendix.

II. The Characteristic Bundle

The previous work [1, 2] has shown that any trivializable gauge field configuration over the Euclidean 4-space $\boldsymbol{E}_4$ contains an important object: the *characteristic vector field* $\hat{\boldsymbol{p}}$. This object governs the geometric and topological properties of such a configuration. Therefore, the purpose of this section aims at a detailed elaboration of all the geometric effects produced by the characteristic vector field. This ultimately will lead us to a Riemann-Cartan space structure.

As does any differentiable section of the tangent bundle $\mathring{\tau}_4$ of $\boldsymbol{E}_4$, the characteristic vector $\hat{\boldsymbol{p}}$ also induces a plane bundle (*characteristic bundle* $\bar{\tau}_4$) over base space $\boldsymbol{E}_4$. However, it is readily found that the connection $\boldsymbol{\Gamma}$ in the characteristic bundle is not determined in general through the SO(3) restriction of the canonical connection $\mathring{\omega}$ of the embedding bundle $\mathring{\tau}_4$. Rather, the original trivializable gauge field $\boldsymbol{A}$, which is the connection in the *representative bundle* $\tilde{\tau}_4$, gives rise to the emergence of that different but closely related connection $\boldsymbol{\Gamma}$ living in the "twin bundle" $\bar{\tau}_4$ associated to $\tilde{\tau}_4$. The interrelationship of both bundles $\tilde{\tau}_4$ and $\bar{\tau}_4$ yields then a complete geometric description of any trivializable gauge field configuration.

The relevant objects in $\bar{\tau}_4$ such as the connection $\boldsymbol{\Gamma}$, its torsion $\boldsymbol{Z}$, the curvature $\boldsymbol{R}$ and the fibre metric $\boldsymbol{G}$ are discussed in detail. The results obtained here are simultaneously preparing the basis for an effective method of solving the Yang-Mills equations below.

1. Trivializability Conditions and Torsion

The torsion emerges very naturally by generalizing the Riemannian connection $\boldsymbol{\Gamma}$ which previously was introduced as the fundamental object for building the Riemannian structure of the dimeron system. For any nondegenerate point $\boldsymbol{x}$ of the trivializable configuration, the extrinsic curvature coefficients $B_{i\lambda}(x)$ represent a basis of $\bar{\Delta}_x$-valued 1-forms for the characteristic 3-plane $\bar{\Delta}_x$ such that their gauge covariant derivative may be expressed by a linear combination in the following way

$$D_\mu B_{i\nu} = \Gamma^\lambda{}_{\nu\mu} B_{i\lambda}\,. \tag{II.1}$$

This may be formally rewritten in the shape of a generally covariant derivative $\mathscr{D}$:

$$\mathscr{D}_\mu B_{i\nu} \equiv D_\mu B_{i\nu} - \Gamma^\lambda{}_{\nu\mu} B_{i\lambda} = 0\,. \tag{II.2}$$

Though we never shall apply such a general (i.e. gauge plus coordinate) transformation, it nevertheless seems rather elegant and effective to work in terms of this non-Riemannian space structure.

Clearly one expects that there will exist severe restrictions upon the $\mathscr{gl}(4, \boldsymbol{R})$ connection $\boldsymbol{\Gamma}$ introduced through equation (II.1). We subsequently shall elaborate these restrictions step by step and finally we will end up with a Riemann-Cartan space structure of constant curvature.

The first restriction upon $\boldsymbol{\Gamma}$ arises from the fact that we are dealing with trivializable gauge fields. The necessary and sufficient conditions for the property of trivializability have been found as

$$\begin{aligned} F_{i\mu\nu} &= -\varepsilon_i{}^{jk} B_{j\mu} B_{k\nu}\,, &\text{(a)}\\ D_\mu B_{i\nu} &= D_\nu B_{i\mu}\,. &\text{(b)} \end{aligned} \tag{II.3}$$

Now, define the torsion $\boldsymbol{Z}$ of $\boldsymbol{\Gamma}$ as usual through [4]

$$Z^\lambda{}_{\mu\nu} = \tfrac{1}{2}(\Gamma^\lambda{}_{\mu\nu} - \Gamma^\lambda{}_{\nu\mu}) \equiv \Gamma^\lambda{}_{[\mu\nu]} \tag{II.4}$$

such that for any gradient vector $\boldsymbol{C}$

$$C_\mu = \partial_\mu C \tag{II.5}$$

the following identity holds [5]

$$\nabla_{[\mu} C_{\nu]} = Z^\lambda{}_{\mu\nu} C_\lambda \tag{II.6}$$

Consequently, using (II.1) for the second trivializability condition (II.3 b), one readily deduces the following constraint upon the torsion $\boldsymbol{Z}$ of $\boldsymbol{\Gamma}$:

$$Z^{\lambda}{}_{\mu\nu} B_{i\lambda} = 0 . \tag{II.7}$$

Since, by its very definition, the characteristic vector $\hat{\boldsymbol{p}}$ is annihilated by the extrinsic curvature 1-forms $\boldsymbol{B}$:

$$B_{i\mu} \hat{p}^{\mu} = 0 , \tag{II.8}$$

one concludes that the torsion always exhibits the following shape

$$Z^{\lambda}{}_{\mu\nu} = -\hat{p}^{\lambda} \hat{p}_{\sigma} Z^{\sigma}{}_{\mu\nu} . \tag{II.9}$$

As a consequence of this special form of the torsion, it effectively will not be present in the definition (II.1) of $\boldsymbol{\Gamma}$. Rather, there emerges only the *essential part* γ of $\boldsymbol{\Gamma}$

$$\gamma^{\lambda}{}_{\mu\nu} = \hat{P}^{\lambda}{}_{\sigma} \Gamma^{\sigma}{}_{\mu\nu} , \tag{II.10}$$

where the projector $\hat{\boldsymbol{P}}$ onto $\bar{\Delta}$ is defined as usual

$$\hat{P}^{\lambda}{}_{\sigma} = g^{\lambda}{}_{\sigma} + \hat{p}^{\lambda} \hat{p}_{\sigma} , \quad \text{(a)}$$
$$\hat{p}^{\lambda} \hat{p}_{\lambda} = g_{\mu\nu} \hat{p}^{\mu} \hat{p}^{\nu} = -1 . \quad \text{(b)} \tag{II.11}$$

Hence, we may equally well rewrite the second condition (II.3 b) as

$$D_{\mu} B_{i\nu} = \gamma^{\lambda}{}_{\nu\mu} B_{i\lambda} , \tag{II.12}$$

where the essential part γ must now be symmetric

$$\gamma^{\lambda}{}_{\mu\nu} = \gamma^{\lambda}{}_{\nu\mu} . \tag{II.13}$$

Obviously, the important point here is that the $gl(4, \boldsymbol{R})$ connection $\boldsymbol{\Gamma}$ is not completely fixed by the gauge potential $\boldsymbol{A}$ and its associated object $\boldsymbol{B}$. The remaining degree of the freedom is its component along $\hat{\boldsymbol{p}}$, the skew-symmetric part of which is just the torsion $\boldsymbol{Z}$ (II.9). This provides us with the possibility of fixing the undetermined component in such a way that the corresponding connection $\boldsymbol{\Gamma}$ will exhibit some desirable features referring to its geometric meaning (see below). But as a consequence, the connection fixed by this procedure will exhibit a non-vanishing torsion $\boldsymbol{Z}$ in general. This means that we cannot hope to deal exclusively with a strictly Riemannian space structure, as was the case for the dimeron configuration.

2. *Bundle Connection*

The geometric meaning of the $gl(4, \boldsymbol{R})$ connection $\boldsymbol{\Gamma}$ is clarified by a closer inspection of the characteristic vector field $\hat{\boldsymbol{p}}$. For the present investigation, we want to leave for a moment the characteristic bundle $\bar{\tau}_4$ and will reconsider the trivializability conditions (II.3) in terms of objects living the representative bundle $\bar{\tau}_4$. Indeed, the conditions (II.3) imply the existence of a (unit) vector field $\hat{n}_{\mu}(x)$, which defines the representative distribution $\bar{\Delta}$, and three special vectors $e^{\mu}{}_{i}(x)$, living within $\bar{\Delta}$ and obeying the following orthonormality relations

$$\hat{n}_{\mu} e^{\mu}{}_{i} = 0 , \quad \text{(a)}$$
$$g_{\mu\nu} e^{\mu}{}_{i} e^{\nu}{}_{j} = g_{ij} . \quad \text{(b)} \tag{II.14}$$

The covariant derivative of anyone of these $\bar{\tau}_4$ sections $e^{\nu}{}_{i}$ may be expressed in terms of the extrinsic curvature of $\bar{\tau}_4$ as

$$D_{\mu} e^{\nu}{}_{i} = -B_{i\mu} \hat{n}^{\nu} . \tag{II.15}$$

Further, it is easy to show that the conditions (II.3) are really satisfied if the objects $\boldsymbol{A}$, $\boldsymbol{B}$ in $\bar{\tau}_4$ are composed of the tetrad vectors as follows

$$A_{i\mu} = \tfrac{1}{2} g_{\lambda\nu} \varepsilon^{jk}{}_{i} e^{\lambda}{}_{j} \partial_{\mu} e^{\nu}{}_{k} , \quad \text{(a)}$$
$$B_{i\mu} = -g_{\lambda\nu} e^{\lambda}{}_{i} \partial_{\mu} \hat{n}^{\nu} . \quad \text{(b)} \tag{II.16}$$

For instance, consider the second condition (II.3 b). Since the normal vector $\hat{n}^{\lambda}$ is a gauge invariant, one readily deduces from (II.16) by use of (II.15)

$$D_{\mu} B_{i\nu} = -g_{\lambda\sigma} e^{\lambda}{}_{i} \partial_{\mu} \partial_{\nu} \hat{n}^{\sigma} \tag{II.17}$$

which immediately implies the second requirement (II.3 b).

Returning to the characteristic vector field, an important conclusion can now be drawn from (II.17). Combining this equation with (II.12) one gets a relationship between the normal $\hat{n}^{\lambda}$ of $\bar{\Delta}$ and the essential part γ of $\boldsymbol{\Gamma}$, the contraction of which with $\hat{\boldsymbol{p}}$ yields

$$\hat{p}^{\nu} \partial_{\nu} \partial_{\mu} \hat{n}^{\lambda} = \gamma^{\sigma}{}_{\mu\nu} \hat{p}^{\nu} \partial_{\sigma} \hat{n}^{\lambda} , \tag{II.18}$$

On the other hand, the very definition of the characteristic vector field implies

$$\hat{P}^{\mu}{}_{\nu} \partial_{\mu} \hat{n}^{\lambda} = \partial_{\nu} \hat{n}^{\lambda} . \tag{II.19}$$

The differentiation of this equation, followed by a contraction with $\hat{p}$, gives a result which looks quite similar as (II.18), namely

$$\hat{p}^{\nu} \partial_{\nu} \partial_{\mu} \hat{n}^{\lambda} = (\partial_{\mu} \hat{P}^{\sigma}{}_{\nu}) \hat{p}^{\nu} (\partial_{\sigma} \hat{n}^{\lambda}) . \tag{II.20}$$

Indeed, the comparison of these two equations gives now our final result

$$\partial_{\lambda} \hat{p}^{\sigma} + \gamma^{\sigma}{}_{\nu\lambda} \hat{p}^{\nu} = 0 . \tag{II.21}$$

This is a rather interesting result because if the essential part γ could be replaced here by the connection $\boldsymbol{\Gamma}$

itself, we would have found that the (Euclidean) unit vector $\hat{\boldsymbol{p}}$ is covariantly constant with respect to the $g\ell(4, \boldsymbol{R})$ connection $\boldsymbol{\Gamma}$, which was introduced through our central equation (II.1) by a quite different reasoning.

Thus the parallel transport ∇ due to $\boldsymbol{\Gamma}$ does not lead tion $\boldsymbol{\Gamma}$ into its essential part γ and some remainder z

$$\Gamma^{\lambda}{}_{\mu\nu} = \gamma^{\lambda}{}_{\mu\nu} + \hat{p}^{\lambda} z_{\mu\nu} \tag{II.22}$$

which is constraint to the condition

$$z_{\mu\nu} \hat{p}^{\mu} = 0 \, . \tag{II.23}$$

By this arrangement, the characteristic vector really becomes covariantly constant

$$\nabla_{\lambda} \hat{p}^{\sigma} \equiv \partial_{\lambda} \hat{p}^{\sigma} + \Gamma^{\sigma}{}_{\nu\lambda} \hat{p}^{\nu} = 0 \tag{II.24}$$

while the torsion assumes the special shape

$$Z^{\lambda}{}_{\mu\nu} = \hat{p}^{\lambda} z_{[\mu\nu]} \, . \tag{II.25}$$

Observe also that the covariant constancy of $\hat{\boldsymbol{p}}$ is consistent with its Euclidean normalization (II.11 b) as well as with the orthogonality relation (II.8) on behalf of the following identity

$$D_{\mu}(B_{i\lambda} \hat{p}^{\lambda}) \equiv (\mathscr{D}_{\mu} B_{i\lambda}) \hat{p}^{\lambda} + B_{i\lambda}(\nabla_{\mu} \hat{p}^{\lambda}) \, . \tag{II.26}$$

At this stage, the geometric meaning of the $g\ell(4, \boldsymbol{R})$ connection $\boldsymbol{\Gamma}$ now becomes evident: $\boldsymbol{\Gamma}$ is to be identified as the (local) connection form in the characteristic bundle $\bar{\tau}_4$. Further, this connection is described from the extrinsic point of view, which is expressed by the reference to the Cartesian coordinates of the embedding space $\boldsymbol{E}_4$. Consider, e.g., some section $V_{\mu}(x)$ of $\bar{\tau}_4$, i.e.

$$\hat{p}^{\mu} V_{\mu} = 0 \, . \tag{II.27}$$

The covariant constancy (II.24) of $\hat{\boldsymbol{p}}$ then guarantees that the derivative of V is also an object living in $\bar{\tau}_4$:

$$\hat{p}^{\mu} \nabla_{\lambda} V_{\mu} = 0 \, . \tag{II.28}$$

Thus the parallel transport ∇ due to $\varLambda$ does not lead off $\bar{\tau}_4$!

Observe, however, that the present connection $\boldsymbol{\Gamma}$ will in general not be identical to the corresponding surface reduction of the canonical connection $\hat{\omega}$ of the embedding space $\boldsymbol{E}_4$. This is readily recognized by considering the following counter example: The t'Hooft-Polyakov monopole solution [2] for a coupled SO(3) Yang-Mills-Higgs system contains a trivializable gauge field, which simultaneously is *static* in the sense that the characteristic lines are straight with respect to $\hat{\omega}$. Hence, the characteristic surfaces are flat 3-sheets (e.g. orthogonal to the x^0-axis). Therefore the surface reduction of $\hat{\omega}$ onto these 3-sheets gives a connection with vanishing curvature which is in contradiction to the non trivial t'Hooft-Polyakov solution for the gauge field! On the other hand, there exist configurations, where both connections $\boldsymbol{\Gamma}$ and $\boldsymbol{A}$ coincide and hence are both SO(3) reductions of $\hat{\omega}$. An example of this is presented by the single meron solutions [1] where both distributions $\bar{\Delta}$ and $\bar{\Delta}$ coincide into the spherical distribution, the integral surfaces of which are concentric 3-spheres around the single meron positions.

3. *Bundle Curvature*

After it has become clear that $\boldsymbol{\Gamma}$ has the meaning of bundle connection in $\bar{\tau}_4$, the next step must consist in looking for the corresponding bundle curvature $\boldsymbol{R}$, whose general definition is, as usual

$$R^{\sigma}{}_{\nu\mu\lambda} = \partial_{\mu} \Gamma^{\sigma}{}_{\nu\lambda} - \partial_{\lambda} \Gamma^{\sigma}{}_{\nu\mu} + \Gamma^{\sigma}{}_{\varkappa\mu} \Gamma^{\varkappa}{}_{\nu\lambda} - \Gamma^{\sigma}{}_{\varkappa\lambda} \Gamma^{\varkappa}{}_{\nu\mu} \, . \tag{II.29}$$

Without knowing the exact shape of $\boldsymbol{\Gamma}$, one nevertheless expects that the curvature $\boldsymbol{R}$ will exhibit some features fitting into the geometric meaning of its connection $\boldsymbol{\Gamma}$.

First, combine the identity

$$\nabla_{[\lambda} \nabla_{\mu]} \hat{p}^{\sigma} = \tfrac{1}{2} R^{\sigma}{}_{\nu\lambda\mu} \hat{p}^{\nu} + Z^{\nu}{}_{\lambda\mu} \nabla_{\nu} \hat{p}^{\sigma} \tag{II.30}$$

with the covariant constancy of $\hat{\boldsymbol{p}}$ (II.24) and find

$$R^{\sigma}{}_{\nu\lambda\mu} \hat{p}^{\nu} = 0 \, . \tag{II.31}$$

While this relation is automatically satisfied, one additionally wants to demand that the bundle curvature 2-form $\boldsymbol{R}$ annihilates the characteristic vector $\hat{\boldsymbol{p}}$ with respect to both entries of the first index pair, i.e.

$$\hat{p}_{\sigma} R^{\sigma}{}_{\nu\lambda\mu} = 0 \, . \tag{II.32}$$

Thus, the curvature $\boldsymbol{R}$ will completely operate within the characteristic distribution $\bar{\Delta}$, in agreement with its meaning as bundle curvature for $\bar{\tau}_4$.

Unfortunately, the requirement (II.32) will not be satisfied in general; rather, it represents a further restriction of the general shape of $\boldsymbol{\Gamma}$. But this just meets with the fact that there still is an undetermined constituent contained in $\boldsymbol{\Gamma}$. Of course, we shall use this freedom in order to adapt $\boldsymbol{\Gamma}$ to the present condition (II.32) in the following way:

Remember first the splitting of $\boldsymbol{\Gamma}$ into its essential part γ and the undetermined remainder z, which is subject merely to the auxiliary condition (II.23). Clearly, one will try now to transform the above condition upon the curvature $\boldsymbol{R}$ into some constraint upon this remainder of $\boldsymbol{\Gamma}$. Indeed, if the decomposition (II.22) of $\boldsymbol{\Gamma}$ is substituted into the curvature $\boldsymbol{R}$ (II.29), the requirement (II.32) is transferred to the desired constraint upon z which then reads as follows

$$\partial_\mu z_{\nu\lambda} - \partial_\lambda z_{\nu\mu} = \gamma^\sigma{}_{\nu\mu}(z_{\sigma\lambda} + \partial_\lambda \hat{p}_\sigma) - \gamma^\sigma{}_{\nu\lambda}(z_{\sigma\mu} + \partial_\mu \hat{p}_\sigma). \tag{II.33}$$

This is a rather complicated differential equation for z, but it admits a very simple solution obeying the auxiliary condition (II.23), namely

$$z_{\sigma\mu} = -\partial_\mu \hat{p}_\sigma . \tag{II.34}$$

This solution may be rewritten as

$$z_{\sigma\mu} = \gamma_{\sigma\nu\mu} \hat{p}^\nu \tag{II.35}$$

if (II.21) is used here. Hence, the final result for the connection $\boldsymbol{\Gamma}$ is

$$\Gamma^\lambda{}_{\mu\nu} = \gamma^\lambda{}_{\mu\nu} + \hat{p}^\lambda \hat{p}^\sigma \gamma_{\mu\sigma\nu} \tag{II.36}$$

and its torsion $\boldsymbol{Z}$ (II.25) becomes

$$Z^\lambda{}_{\mu\nu} = \hat{p}^\lambda \partial_{[\mu} \hat{p}_{\nu]} . \tag{II.37}$$

The special $\boldsymbol{\Gamma}$ (II.36) as a solution to the problem (II.32) obviously is always possible and hence is referred to as the *standard connection* hereafter. Since it is based upon a particular solution of the differential equation for z (II.33), there might exist further connections solving the problem (II.32). Especially, one could think of solutions $\boldsymbol{\Gamma}$ the torsion of which vanishes. We shall return to this point later.

Further, let us remark that the standard connexion $\boldsymbol{\Gamma}$ (II.36) owns an additional, highly welcome property. This consists in the fact that it makes the covariant vector constant, too; i.e.

$$\nabla_\lambda \hat{p}_\sigma \equiv \partial_\lambda \hat{p}_\sigma - \Gamma^\varrho{}_{\sigma\lambda} \hat{p}_\varrho = 0 . \tag{II.38}$$

Observe here, that at the present stage we still raise and lower the indices by means of the Euclidean metric $g_{\mu\nu}$, i.e.

$$\hat{p}_\sigma = g_{\sigma\lambda} \hat{p}^\lambda , \quad \gamma_{\mu\sigma\nu} = g_{\mu\lambda} \gamma^\lambda{}_{\sigma\nu} , \text{ etc.} \tag{II.39}$$

Surely, the Euclidean metric $\boldsymbol{g}$ is not covariantly constant with respect to the standard connection $\boldsymbol{\Gamma}$ (II.36), but nevertheless both equations (II.24) and (II.38) are holding simultaneously. This strongly suggests the existence of some metric $\boldsymbol{G}$ which is covariantly constant but acts just as the Euclidean $\boldsymbol{g}$ when applied to vectors pointing into the characteristic direction, i.e.

$$G_{\mu\nu} \hat{p}^\mu \hat{p}^\nu = g_{\mu\nu} \hat{p}^\mu \hat{p}^\nu = -1 . \tag{II.40}$$

We are going to verify this supposition immediately.

4. *Fibre Metric*

In order to complete the characteristic geometry, we need a fibre metric for $\bar{\tau}_4$. Remember that the fibre metric for the representative bundle $\bar{\tau}_4$ is just the 3-dimensional Euclidean $\boldsymbol{g}$ which is covariantly constant with respect to the trivializable connection $\boldsymbol{A}$ in $\bar{\tau}_4$:

$$D_\lambda g_{ij} = 0 . \tag{II.41}$$

Similarly, we try to find some fibre metric $\boldsymbol{G}$ for the tangent bundle $\mathring{\tau}_4$ of base space $\boldsymbol{E}_4$, which on the one hand obeys

$$\nabla_\lambda G_{\mu\nu} = 0 , \tag{II.42}$$

and on the other hand respects the condition (II.40). The restriction $\bar{\boldsymbol{G}}$ of $\boldsymbol{G}$ onto $\bar{\Delta}$ may then serve as the fibre metric in the characteristic bundle $\bar{\tau}_4$:

$$\bar{G}_{\mu\nu} = G_{\mu\nu} + \hat{p}_\mu \hat{p}_\nu . \tag{II.43}$$

Observe that the constancy of the fibre metric $\bar{\boldsymbol{G}}$ is guaranteed by (II.38) and (II.42):

$$\nabla_\lambda \bar{G}_{\mu\nu} = 0 . \tag{II.44}$$

A solution to this problem is readily at hand, namely [6]

$$\bar{G}_{\mu\nu} = c^2 B_{i\mu} B^i{}_\nu \equiv c^2 B_{\mu\nu} . \tag{II.45}$$

The constancy condition (II.44) is immediately verified since

$$\nabla_\lambda B_{\mu\nu} = 0 \tag{II.46}$$

by simply observing the identity

$$\nabla_\lambda B_{\mu\nu} = (\mathscr{D}_\lambda B_{i\mu}) B^i{}_\nu + B_{i\mu} (\mathscr{D}_\lambda B^i{}_\nu) \tag{II.47}$$

together with the general constancy condition for the extrinsic curvature $\boldsymbol{B}$ (II.2).

Besides the Euclidean $\boldsymbol{g}$, we now have a second metric $\boldsymbol{G}$ in the tangent bundle $\mathring{\tau}_4$ and hence we must be somewhat cautious when raising and lowering the Cartesian indices. We apply the following convention: whenever an index is moved from its original position by use of the non Euclidean $\boldsymbol{G}$, we indicate this by a dot. The following exemplifying relations have to be understood in this sense

$$
\begin{aligned}
\hat{p}_{\dot{\mu}} &= G_{\mu\nu}\,\hat{p}^{\nu} = \hat{\mathrm{p}}_{\mu} && \text{(a)}\\
G^{\mu\nu} &= c^2\,B^{\mu\nu} - \hat{p}^{\mu}\,\hat{p}^{\nu}, && \text{(b)}\\
\mathrm{G}^{\dot{\mu}\dot{\nu}} &= c^{-2}(B^{-1})^{\mu\nu} - \hat{p}^{\mu}\,\hat{p}^{\nu}, && \text{(c)}\\
(B^{-1})^{\mu\nu}\,B_{\nu\lambda} &= \hat{P}^{\mu}{}_{\lambda}, && \text{(d)}\\
G^{\dot{\mu}\dot{\nu}}\,G_{\nu\lambda} &= g^{\mu}{}_{\lambda}. && \text{(e)}
\end{aligned}
\qquad \text{(II.48)}
$$

Observe here, that in general the objects $G^{\mu\nu}$, $B^{\mu\nu}$, $\hat{P}_{\mu\nu}$, $\hat{P}^{\mu\nu}$ are not constant, rather we will have

$$
\begin{aligned}
\nabla_{\lambda}\,G^{\dot{\mu}\dot{\nu}} &= 0, && \text{(a)}\\
\nabla_{\lambda}(B^{-1})^{\mu\nu} &= 0, && \text{(b)}\\
\nabla_{\lambda}\,\hat{P}^{\mu}{}_{\nu} &= 0. && \text{(c)}
\end{aligned}
\qquad \text{(II.49)}
$$

As is well known, the existence of a metric $\boldsymbol{G}$ also has some consequences for the curvature $\boldsymbol{R}$. For example, consider the general identity

$$
[\nabla_{\lambda}\,\nabla_{\sigma} - \nabla_{\sigma}\,\nabla_{\lambda}]\,G_{\mu\nu}
\equiv -R^{\varrho}{}_{\mu\lambda\sigma}\,G_{\varrho\nu} - R^{\varrho}{}_{\nu\lambda\sigma}\,G_{\mu\varrho} + 2\,Z^{\varrho}{}_{\lambda\sigma}\,\nabla_{\varrho}\,G_{\mu\nu} \qquad \text{(II.50)}
$$

and observe here that $\boldsymbol{G}$ is constant (II.42). This readily yields the skew-symmetry for the curvature operator $\boldsymbol{R}$

$$
R_{\dot{\nu}\mu\lambda\sigma} = -R_{\dot{\mu}\nu\lambda\sigma}\,. \qquad \text{(II.51)}
$$

So we see that the geometry of the characteristic bundle essentially is what is called *Riemann-Cartan structure* [3]. The connection $\boldsymbol{\Gamma}$ in a Riemann-Cartan space can be decomposed into its Riemannian part $\tilde{\boldsymbol{\Gamma}}$ and the contorsion $\boldsymbol{K}$ such that

$$
\Gamma^{\lambda}{}_{\mu\nu} = \tilde{\Gamma}^{\lambda}_{\mu\nu} + K^{\lambda}{}_{\mu\nu}\,. \qquad \text{(II.52)}
$$

Here, the Riemannian part $\tilde{\boldsymbol{\Gamma}}$ is given by the well known Christoffel symbols

$$
\tilde{\Gamma}^{\lambda}{}_{\mu\nu} = \tfrac{1}{2}\,G^{\dot{\lambda}\dot{\sigma}}(\partial_{\mu}\,G_{\sigma\nu} + \partial_{\nu}\,G_{\sigma\mu} - \partial_{\sigma}\,G_{\mu\nu}), \qquad \text{(II.53)}
$$

whereas the contorsion $\boldsymbol{K}$ is composed of the torsion $\boldsymbol{Z}$ in the following way

$$
K^{\lambda}{}_{\mu\nu} = Z^{\lambda}{}_{\mu\nu} + Z_{\dot{\mu}\nu}{}^{\dot{\lambda}} + Z_{\dot{\nu}\mu}{}^{\dot{\lambda}}. \qquad \text{(II.54)}
$$

III. Bundle Isomorphism

The relationship of the "twin bundles" $\bar{\tau}_4$ and $\overline{\tau}_4$ is closer than it might appear through the preceding considerations. In fact, these two bundles are revealed to be isomorphic and hence may be looked upon as being identical. This means that the geometric objects in $\overline{\tau}_4$, such as connection $\boldsymbol{\Gamma}$, curvature $\boldsymbol{R}$ and fibre metric $\overline{\boldsymbol{G}}$, are the Cartesian version of the corresponding intrinsic objects in $\bar{\tau}_4$.

Subsequently, the bundle isomorphism is worked out in detail. The results are then used in order to express the curvature $\boldsymbol{R}$ of $\overline{\tau}_4$ in terms of the extrinsic curvature $\mathbf{B}$ of $\bar{\tau}_4$. This yields the generalization of the previous dimeron results. Surprisingly enough, most of the particular features of the dimeron case are surviving the generalization process. For instance, the general Riemann-Cartan structure formally meets with the relations defining a conformally flat, locally symmetric Einstein space. As a consequence of this fact, the characteristic surfaces are equipped with a geometry of constant curvature!

1. Bundle Map

The two bundles $\bar{\tau}_4$ and $\overline{\tau}_4$ over the same base space $\boldsymbol{E}_4$ are mutually interrelated by a map which is induced by the extrinsic curvature $\boldsymbol{B}$ of $\bar{\tau}_4$. Consider first the bundle map $[\overline{B}]$

$$
[B]:\quad \bar{\tau}_4 \to \overline{\tau}_4\,. \qquad \text{(III.1)}
$$

By this map any section $\boldsymbol{v} \in \bar{\tau}_4$ is cast into its image $\boldsymbol{V} \in \overline{\tau}_4$ such that

$$
V_{\mu} = c\,B_{i\mu}\,v^{i}\,. \qquad \text{(III.2)}
$$

Here, we have included some arbitrary length parameter c for dimensional reasons. Clearly such a bundle map $[\overline{B}]$ extends to all the tensor products. For instance, the fibre metric $\overline{\boldsymbol{G}}$ (II.45) of $\overline{\tau}_4$ is revealed by the present approach as the image of the fibre metric $\boldsymbol{g}$ (II.41) of $\bar{\tau}_4$:

$$
B_{\mu\nu} = B_{i\mu}\,B_{j\nu}\,g^{ij}\,. \qquad \text{(III.3)}
$$

As a consequence, the scalar product is an invariant with respect to $[\overline{B}]$ for any two sections $(\boldsymbol{u}, \boldsymbol{v}) \in \bar{\tau}_4$ and their images $(\boldsymbol{U}, \boldsymbol{V}) \in \overline{\tau}_4$,

$$
\boldsymbol{G}(\boldsymbol{U}, \boldsymbol{V}) = \boldsymbol{g}(\boldsymbol{u}, \boldsymbol{v}), \qquad \text{(III.4)}
$$

in components

$$
G^{\dot{\mu}\dot{\nu}}\,U_{\mu}\,V_{\nu} = g_{ij}\,v^{i}\,u^{j}\,. \qquad \text{(III.5)}
$$

The inverse map of $[\overline{B}]$ is $[\bar{B}]$:

$$
[\bar{B}]:\quad \overline{\tau}_4 \to \bar{\tau}_4\,. \qquad \text{(III.6)}
$$

The inverse map $[\bar{B}]$ takes any section $V \in \bar{\tau}_4$ into $\bar{\tau}_4$ such that

$$v_i = c\,B_{i\mu}\,V^{\mu}. \tag{III.7}$$

Combining the two maps, one readily deduces from (III.2) and (III.7) that the composition really gives the identity map for the corresponding image spaces

$$\begin{aligned}[\bar{B}] \circ [\bar{B}] &= \mathrm{id}_{\bar{\tau}_4}, \quad \text{(a)}\\ [\bar{B}] \circ [\bar{B}] &= \mathrm{id}_{\bar{\tau}_4}. \quad \text{(b)}\end{aligned} \tag{III.8}$$

This verifies the one-to-one correspondence of both bundles $\bar{\tau}_4$ and $\bar{\tau}_4$!

2. Extrinsic Versus Intrinsic Curvature

For an affine bundle isomorphism [5], it is not enough that the fibres over any point $\boldsymbol{x}$ of the base space are in one-to-one correspondence. Additionally, it is necessary that the splitting into horizontal and vertical subspaces is consistent with the bundle map. In other words, the process of parallel displacement must commute with the bundle map. But this is just guaranteed, if the covariant derivatives are commuting with the bundle map, i.e. we must have

$$\begin{aligned}\nabla \circ [\bar{B}] &= [\bar{B}] \circ D, \quad \text{(a)}\\ D \circ [\bar{B}] &= [\bar{B}] \circ \nabla. \quad \text{(b)}\end{aligned} \tag{III.9}$$

In components, these requirements read

$$\begin{aligned}\nabla_\lambda V_\mu &= c\,B_{i\mu}(D_\lambda v^i), \quad \text{(a)}\\ D_\lambda v_i &= c\,B_{i\mu}\nabla_\lambda V^{\mu}. \quad \text{(b)}\end{aligned} \tag{III.10}$$

However, this is just satisfied by differentiating the corresponding bundle maps (III.2) and (III.7), which first yields the following identities

$$\begin{aligned}\nabla_\lambda V_\mu &\equiv c(\mathcal{D}_\lambda B_{i\mu})\,v^i + c\,B_{i\mu}(D_\lambda v^i), \quad \text{(a)}\\ D_\lambda v_i &\equiv c(\mathcal{D}_\lambda B_{i\mu})\,V^{\mu} + c\,B_{i\mu}(\nabla_\lambda V^{\mu}). \quad \text{(b)}\end{aligned} \tag{III.11}$$

Next, one observes the constancy condition (II.2) for the extrinsic curvature $\boldsymbol{B}$ and then readily verifies the requirements (III.10).

By these arguments, the extrinsic curvature coefficients are recognized as mixed objects living partially in $\bar{\tau}_4$ and $\bar{\tau}_4$. This is the reason why these objects must be parallel transported by the simultaneous use of the trivializable connection $\boldsymbol{A}$ and the standard connection $\boldsymbol{\Gamma}$. Observe, that the covariant derivative $\mathcal{D}$ (II.2) respects both transport laws and obviously applies to all mixed objects of arbitrary rank. Thus the previous transport laws $\boldsymbol{D}$ and $\boldsymbol{V}$ are revealed as special cases of the more general $\mathcal{D}$.

As an application of the bundle isomorphism, we can look now for a relationship between both bundle curvatures. Since the intrinsic curvature $\boldsymbol{F}$ in $\bar{\tau}_4$ is completely fixed by the extrinsic curvature $\boldsymbol{B}$ via the first trivializability condition (II.3 a), one expects that the curvature $\boldsymbol{R}$ of $\bar{\tau}_4$ may be expressed in terms of $\boldsymbol{B}$, too. Indeed, the desired link of $\boldsymbol{R}$ and $\boldsymbol{B}$ is readily obtained by differentiating once more the relations (III.9) which in the abstract language yields

$$\begin{aligned}\nabla \circ \nabla \circ [\bar{B}] &= [\bar{B}] \circ \mathcal{D} \circ D, \quad \text{(a)}\\ \mathcal{D} \circ D \circ [\bar{B}] &= [\bar{B}] \circ \nabla \circ \nabla, \quad \text{(b)}\end{aligned} \tag{III.12}$$

or, more concretely and after skew-symmetrization,

$$\begin{aligned}\nabla_{[\sigma}\nabla_{\lambda]}\,V_\mu &= c\,B_{i\mu}\,\mathcal{D}_{[\sigma}D_{\lambda]}\,v^i, \quad \text{(a)}\\ \mathcal{D}_{[\sigma}D_{\lambda]}\,v_i &= c\,B_{i\mu}\,\nabla_{[\sigma}\nabla_{\lambda]}\,V^{\mu}. \quad \text{(b)}\end{aligned} \tag{III.13}$$

Further, we have to observe here the identities

$$\mathcal{D}_{[\sigma}\,D_{\lambda]}\,v^i \equiv \tfrac{1}{2}\,\varepsilon^{ij}{}_k\,F_{j\sigma\lambda}\,v^k + Z^{\varrho}{}_{\sigma\lambda}(D_\varrho\,v^i) \tag{III.14}$$

and

$$\nabla_{[\sigma}\,\nabla_{\lambda]}\,V_\mu \equiv -\tfrac{1}{2}\,R^{\varrho}{}_{\mu\sigma\lambda}\,V_\varrho + Z^{\varrho}{}_{\sigma\lambda}(\nabla_\varrho\,V_\mu). \tag{III.15}$$

Remembering now that $\boldsymbol{V}$ and $\boldsymbol{v}$ are linked by the bundle maps (III.2) and (III.7), we finally find from (III.13) the desired result

$$R^{\varrho}{}_{\mu\sigma\lambda} = -[\hat{P}^{\varrho}{}_{\sigma}\,B_{\mu\lambda} - \hat{P}^{\varrho}{}_{\lambda}\,B_{\mu\sigma}]. \tag{III.16}$$

This special shape of the curvature $\boldsymbol{R}$ has already been encountered previously for the particular case of the di-meron configuration! It is also instructive to have a look on the totally covariant curvature tensor which is obtained from the mixed one (III.16) by lowering the first index by means of the fibre metric $\bar{\boldsymbol{G}}$ (II.45):

$$R_{\mu\nu\sigma\lambda} = -c^2\,[B_{\mu\sigma}\,B_{\nu\lambda} - B_{\mu\lambda}\,B_{\nu\sigma}]. \tag{III.17}$$

In this form, the curvature $\boldsymbol{R}$ of $\bar{\tau}_4$ is completely expressed in terms of the extrinsic curvature $\boldsymbol{B}$ of $\bar{\tau}_4$. Similarly, one may raise the second index in (III.16) yielding a shape of $\boldsymbol{R}$ which exclusively contains the projector $\hat{\boldsymbol{P}}$:

$$R^{\mu\nu}{}_{\sigma\lambda} = -c^{-2}\,[\hat{P}^{\mu}{}_{\sigma}\,\hat{P}^{\nu}{}_{\lambda} - \hat{P}^{\mu}{}_{\lambda}\,\hat{P}^{\nu}{}_{\sigma}]. \tag{III.18}$$

Evidently, the Riemannian $\boldsymbol{R}$ is consistent with the covariant constancy of the characteristic vector (cf. (II.31) and (II.32)), but it simultaneously forbids the existence of a second constant vector besides $\hat{\boldsymbol{p}}$. More generally, the existence of a $\bar{\tau}_4$ section $\boldsymbol{U}$ is forbidden the derivative of which looks as follows

$$\nabla_\lambda U_\mu = N_\mu \hat{p}_\lambda . \tag{III.19}$$

In order to prove this assertion, evoke the identity (III.15) together with the result (III.16) for the Riemannian and find

$$\nabla_{[\sigma} \nabla_{\lambda]} U_\mu = \hat{p}_{[\lambda} \nabla_{\sigma]} N_\mu = B_{\mu[\lambda} U_{\sigma]} . \tag{III.20}$$

Here, the right-hand side does not contain $\hat{\boldsymbol{p}}$ and hence it must vanish which is possible only for vanishing $\boldsymbol{U}$. Therefore (III.19) does not admit a non-trivial solution for $\boldsymbol{U}$ and thus there is no covariantly constant section in $\bar{\tau}_4$.

As a consequence, the general shape of the covariant derivative of any $\bar{\tau}_4$ section $\boldsymbol{V}$ reads

$$\nabla_\lambda V_\mu = N_\mu \hat{p}_\lambda + M_{\mu\lambda} , \tag{III.21}$$

where the previous constraint (II.28) yields, without loss of generality

$$\hat{p}^\mu N_\mu = \hat{p}^\mu M_{\mu\lambda} = \hat{p}^\lambda M_{\mu\lambda} = 0 . \tag{III.22}$$

Observe here that, in contrast to $\boldsymbol{N}$, the tensor $\boldsymbol{M}$ is not allowed to be zero. In order to give a nice demonstration for this effect, resolve the constancy condition (II.2) for the extrinsic curvature with respect to the coordinate covariant derivative $\boldsymbol{\nabla B}$ and find

$$\nabla_\lambda B_{i\mu} = N_{i\mu} \hat{p}_\lambda + M_{i\mu\lambda} , \tag{III.23}$$

where

$$\begin{aligned} N_{i\mu} &= \varepsilon_i{}^{jk} B_{k\mu} (\hat{p}^\sigma A_{j\sigma}) , && \text{(a)} \\ M_{i\mu\lambda} &= -\varepsilon_i{}^{jk} B_{k\mu} A_{j\sigma} \hat{P}^\sigma{}_\lambda . && \text{(b)} \end{aligned} \tag{III.24}$$

Now consider the positive gauge ($\boldsymbol{A} = \boldsymbol{B}$), where (III.23) is reduced to

$$\nabla_\lambda B_{i\mu} = F_{i\lambda\mu} . \tag{III.25}$$

This clearly demonstrates that the vector $\boldsymbol{N}$, but not the tensor $\boldsymbol{M}$, may be gauged off in (III.23).

By a similar argument one can show that there exists no constant, second rank object in the tensor bundle associated to $\bar{\tau}_4$ besides the fibre metric $\bar{G}_{\mu\nu}$.

3. Generalized Conformality

The preceding results of the bundle isomorphism can also be used to see in what way the conformal properties of the dimeron geometry may be generalized to an arbitrary configuration. To this end, remember that the necessary and sufficient conditions for a Riemannian space ($n \geq 4$) being conformally flat consist in the requirement of a vanishing Weyl tensor $\boldsymbol{W}$. The latter one is given for any space dimension (n) through [7]

$$W^{\nu\dot{\mu}}{}_{\varkappa\lambda} = R^{\nu\dot{\mu}}{}_{\varkappa\lambda} - \frac{2}{n-2} (\mathscr{R}^{\dot{\nu}}{}_{[\varkappa} G^{\dot{\mu}}{}_{\lambda]} - \mathscr{R}^{\dot{\mu}}{}_{[\varkappa} G^{\dot{\nu}}{}_{\lambda]}) + \frac{2}{(n-1)(n-2)} \mathscr{S} G^{\dot{\nu}}{}_{[\varkappa} G^{\dot{\mu}}{}_{\lambda]} . \tag{III.26}$$

Strictly speaking, this condition of vanishingWeyl tensor is only valid for a Riemannian space, but it is suggestive to transfer this criterium formally to any space endowed with a symmetric metric $\boldsymbol{G}$ and unique Ricci tensor $\mathscr{R}$

$$\mathscr{R}^{\dot{\mu}}{}_\nu = R^{\sigma\dot{\mu}}{}_{\sigma\nu} = R^{\mu\dot{\sigma}}{}_{\nu\sigma} . \tag{III.27}$$

Obviously, the deviation of such a generalized, conformally flat space from an ordinary Riemannian space is in some sense "minimal" (an "almost Riemannian" space).

Applying this generalized definition to the present situation, it is readily recognized that the bundle geometry of a general trivializable gauge field configuration is indeed due to a conformally flat Riemann-Cartan structure: First, contract the curvature tensor $\boldsymbol{R}$ (III.18) and find for the Ricci $\mathscr{R}$ (III.27)

$$\begin{aligned} \mathscr{R}^{\dot{\mu}}{}_\sigma &= -2c^{-2} \hat{P}^\mu{}_\sigma , && \text{(a)} \\ \mathscr{R}_{\mu\sigma} &= -2 B_{\mu\sigma} . && \text{(b)} \end{aligned} \tag{III.28}$$

Then obtain the curvature scalar $\mathscr{S}$ as

$$\mathscr{S} \equiv \mathscr{R}^{\dot{\mu}}{}_\mu = -6c^{-2} \tag{III.29}$$

which clearly must be constant because of the general constancy of the Riemannian $\boldsymbol{R}$ (III.17) and the Ricci $\mathscr{R}$ (III.28):

$$\nabla_\lambda R_{\dot{\mu}\nu\varrho\sigma} = \nabla_\lambda \mathscr{R}_{\mu\nu} = 0 . \tag{III.30}$$

Next, express the Ricci $\mathscr{R}$ in terms of the scalar $\mathscr{S}$ as

$$\mathscr{R}_{\mu\sigma} = \tfrac{1}{3} \mathscr{S} G_{\mu\sigma} \tag{III.31}$$

which says that we are essentially dealing with an Einstein space structure of dimension $n = 3$. The appearance of this particular space dimension seems plausible if we remember that the bundle geometry is based upon the 3-dimensional distribution Δ.

Now we can alternatively use for the Weylian $\boldsymbol{W}$ (III.26) the full metric $\boldsymbol{G}$, together with dimension $n = 4$; or we may use the fibre metric $\boldsymbol{G}$ and must then choose $n = 3$. In both cases we find that the Weylian vanishes which proves our assertion referring to the generalized conformal flatness.

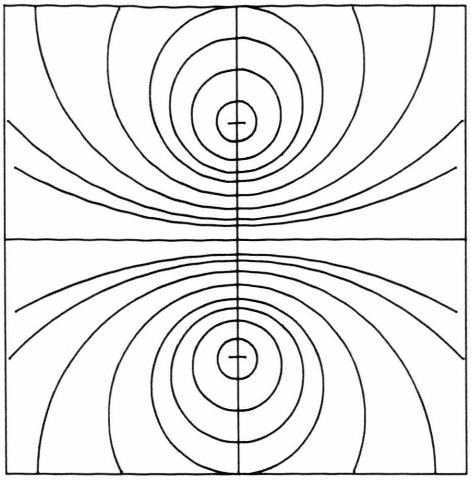

Fig. 1. The characteristic surfaces of the dimeron solution are the 3-spheres $\Phi(x) = \text{const}$ (cf. (V.29)). If the meron centers are fixed at $x^0 = a, b$ resp., then the centers of the characteristic surfaces with radius Φ^{-1} are found at

$$x^0 = \frac{a+b}{2} \pm \frac{a-b}{2}\left(1 + \left(\frac{a-b}{2}\Phi\right)^{-2}\right)^{1/2}.$$

There is a theorem in Riemannian geometry which says that any Einstein space, which is conformally flat, is a space of constant curvature. The generalization of this theorem to the present non-Riemannian structure is satisfied evidently: we have an Einstein-space structure of dimension $n = 3$ (cf. (III.31)), the Weylian of which vanishes as well in the intrinsic ($n = 3$) as in the extrinsic ($n = 4$) approach so that the generalized conformality of the Einstein space appears to be ensured. However, there is a somewhat delicate point here. The vanishing of the Weylian is a sufficient condition for the conformal flatness only for dimensions $n \geqq 4$. But our bundle geometry really refers to a 3-dimensional space; consequently we have to evoke a further criterium for the conformal flatness [8]. This consists in the vanishing of the tensor $\mathscr{W}$

$$\mathscr{W}_{\mu\nu\lambda} = \nabla_\lambda \mathscr{R}_{\mu\nu} - \nabla_\nu \mathscr{R}_{\mu\lambda} + \tfrac{1}{4}(G_{\mu\lambda}\,\partial_\nu \mathscr{S} - G_{\mu\nu}\,\partial_\lambda \mathscr{S}). \qquad \text{(III.32)}$$

But, the vanishing of this tensor $\mathscr{W}$ evidently is guaranteed by the constancy of the curvature scalar $\mathscr{S}$ and of the Ricci $\mathscr{R}$. This completes the proof for the "conformal flatness" of the characteristic bundle geometry.

On the other hand, the curvature tensor $\boldsymbol{R}$ is just of the form required for a space of constant curvature when it is expressed in terms of the fibre metric $\bar{\boldsymbol{G}}$:

$$R_{\mu\nu\sigma\lambda} = -c^{-2}[\bar G_{\mu\sigma}\,\bar G_{\nu\lambda} - \bar G_{\mu\lambda}\,\bar G_{\nu\sigma}]. \qquad \text{(III.33)}$$

Hence, the above theorem also holds for the present non-Riemannian structure. As a result, the characteristic surfaces (if they exist) emerge here as 3-dimensional submanifolds of the base space, which are equipped with a Riemannian geometry of constant curvature! A nice demonstration for this effect is represented again by the di-meron configuration: the dimeron surfaces are 3-spheres and hence are equipped naturally with a constant curvature geometry via their embedding into the base space $\boldsymbol{E}_4$ (Figure 1). But this natural geometry, inherited by projection from the embedding Euclidean geometry, is not identical to the characteristic bundle geometry; nevertheless both geometries exhibit the property of constant curvature! On the other hand, both geometries on the dimeron surfaces really become icdentical, if one of the two meron centers is removed to infinity (see the discussion below (II.28)).

IV. Integrability and Torsion

The bundle isomorphism has made clear that the geometric properties of the twin bundles $\tilde\tau_4$ and $\bar\tau_4$ are essentially the same. At first sight, this might appear as a paradoxical situation: On the one hand, a general theorem [9] ensures that the geometry of $\tilde\tau_4$ must be free of torsion because it is the subgeometry due to the reduction of the trivial embedding bundle $\mathring{\tau}_4$ which itself naturally has vanishing torsion. On the other hand, we could not prevent the torsion from penetrating into the bundle geometry of $\bar\tau_4$, if the latter one is based upon the standard connection $\boldsymbol{\Gamma}$.

Thus the question arises: under what conditions can one get rid of torsion again? We shall show that the integrability of the characteristic distribution $\bar\Delta$ is a sufficient condition. This means that whenever the integrability condition is satisfied, one can use a strictly Riemannian connection $\overset{*}{\boldsymbol{\Gamma}}$ *(characteristic connection)* in place of the standard connection $\boldsymbol{\Gamma}$. The corresponding Riemannian metric $\overset{*}{\boldsymbol{G}}$ *(characteristic metric)* can be found such that $\overset{*}{\boldsymbol{\Gamma}}$ is the Levi-Civita connection of $\overset{*}{\boldsymbol{G}}$. The Riemannian structure found in this way is always of the conformally flat and symmetric type.

1. *Frobenius Condition*

In a Riemann-Cartan structure there are several identities involving the torsion $\boldsymbol{Z}$. For the present case these identities are assuming a very particular form due to the fact that we are concerned with an almost

Riemannian structure. If this special structure is complemented by the Frobenius integrability condition for the characteristic distribution $\bar{\Delta}$, the torsion aquires a suggestive shape which gives us a hint how to eliminate it completely.

First consider the second Bianchi identity

$$R^{\lambda}{}_{\sigma[\mu\nu;\varrho]} = 2R^{\lambda}{}_{\sigma\varkappa[\varrho}Z^{\varkappa}{}_{\mu\nu]}. \tag{IV.1}$$

One is easily convinced that this identity is satisfied trivially here because the left hand side vanishes on account of the covariant constancy of curvature (III.30). On the other hand, the right hand-side is zero, too, because the curvature 2-form annihilates the characteristic vector $\hat{\boldsymbol{p}}$ (cf. (II.31, 32) and (III.16)) and consequently also annihilates the torsion $\boldsymbol{Z}$ according to (II.9). So the second Bianchi identity gives no further restriction.

But now consider the first Bianchi identity

$$-R^{\varrho}{}_{[\mu\nu\lambda]} = 2\nabla_{[\mu}Z^{\varrho}{}_{\nu\lambda]} + 4Z^{\varkappa}{}_{[\mu\nu}Z^{\varrho}{}_{\lambda]\varkappa}. \tag{IV.2}$$

The left-hand side vanishes again because the curvature tensor exhibits all the index symmetry properties of an ordinary Riemannian. Therefore, the torsion must obey the following identity

$$\nabla_{[\mu}Z^{\varrho}{}_{\nu\lambda]} + 2Z^{\varkappa}{}_{[\mu\lambda}Z^{\varrho}{}_{\lambda]\varkappa} = 0. \tag{IV.3}$$

This identity is simplified further if the characteristic distribution is integrable. The Frobenius integrability condition upon the normal $\hat{\boldsymbol{p}}$ of $\bar{\Delta}$ is [10]

$$\partial_{[\mu}\hat{p}_{\lambda]} = f_{[\lambda}\hat{p}_{\mu]}. \tag{IV.4}$$

Without loss of generality, we may assume here

$$f^{\mu}\hat{p}_{\mu} = 0, \tag{IV.5}$$

and thus the torsion of the standard connection $\boldsymbol{\Gamma}$ acquires the following special shape according to (II.37)

$$Z^{\lambda}{}_{\mu\nu} = \hat{p}^{\lambda}\hat{p}_{[\mu}f_{\nu]}. \tag{IV.6}$$

As a consequence, the non-linear terms occurring in the identity (IV.3) cancel and we are left with

$$\nabla_{[\mu}Z^{\varrho}{}_{\nu\lambda]} = 0. \tag{IV.7}$$

A further consequence of the particular torsion $\boldsymbol{Z}$ (IV.6) is the following relation:

$$Z^{\lambda}{}_{\mu\nu} + Z_{\dot{\nu}}{}^{\dot{\lambda}}{}_{\mu} + Z_{\dot{\mu}\nu}{}^{\dot{\lambda}} = 0. \tag{IV.8}$$

For this reason, the contorsion $\boldsymbol{K}$ (II.54) is simplified into

$$K^{\lambda}{}_{\mu\nu} = 2Z_{\dot{\nu}\mu}{}^{\dot{\lambda}}. \tag{IV.9}$$

So we see that the integrability of $\bar{\Delta}$ produces a very special Riemann-Cartan structure which comes very close to a strictly Riemannian geometry. The latter fact is seen more clearly if the motion of a "test particle" on this geometrical background is considered. For a Riemannian-Cartan space, the question of particle motion leads to a certain ambiguity [3]: which is the correct path to be followed by a test particle? Is it an autoparallel or a geodesic curve? For a strictly Riemannian space, both types of curves coincide and there is no problem. The ambiguity clearly persists also for the present Riemann-Cartan geometry albeit in a less severe form. Remember that, along an autoparallel curve, the (unit) tangent vector $\boldsymbol{t}$ is kept parallel with respect to the (standard) connection $\boldsymbol{\Gamma}$

$$t^{\lambda}\nabla_{\lambda}t^{\mu} = 0 \quad (t^{\lambda} \equiv \mathrm{d}x^{\lambda}/\mathrm{d}s). \tag{IV.10}$$

A similar statement holds for a geodesic curve, too; but here the parallel transport refers to the Riemannian part $\tilde{\boldsymbol{\Gamma}}$ of $\boldsymbol{\Gamma}$ (cf. (II.52)):

$$t^{\lambda}\tilde{\nabla}_{\lambda}t^{\mu} = 0. \tag{IV.11}$$

The corresponding differential equations for both types of curves are found from (IV.10, 11) as

$$\frac{\mathrm{D}^2x^{\mu}}{\mathrm{d}s^2} \equiv \frac{\mathrm{d}^2x^{\mu}}{\mathrm{d}s^2} + \Gamma^{\mu}{}_{\nu\lambda}\frac{\mathrm{d}x^{\nu}}{\mathrm{d}s}\frac{\mathrm{d}x^{\lambda}}{\mathrm{d}s} = 0, \quad \text{(a)}$$

$$\frac{\tilde{\mathrm{D}}^2x^{\mu}}{\mathrm{d}s^2} \equiv \frac{\mathrm{d}^2x^{\mu}}{\mathrm{d}s^2} + \tilde{\Gamma}^{\mu}{}_{\nu\lambda}\frac{\mathrm{d}x^{\nu}}{\mathrm{d}s}\frac{\mathrm{d}x^{\lambda}}{\mathrm{d}s} = 0. \quad \text{(b)} \tag{IV.12}$$

The solution of these equations of motion will be found to be quite different in the general case (Appendix A).

However in the present geometric situation, being characterized through (IV.6) and (IV.9), the covariant differential operators occurring in (IV.12) are differing by a single torsion term

$$\frac{\mathrm{D}^2x^{\mu}}{\mathrm{d}s^2} = \frac{\tilde{\mathrm{D}}^2x^{\mu}}{\mathrm{d}s^2} + 2Z_{\dot{\lambda}\nu}{}^{\dot{\mu}}\frac{\mathrm{d}x^{\nu}}{\mathrm{d}s}\frac{\mathrm{d}x^{\lambda}}{\mathrm{d}s} = 0. \tag{IV.13}$$

Therefore, the two types of motion coincide whenever the following requirement is satisfied

$$Z_{\dot{\lambda}\nu}{}^{\dot{\mu}}t^{\nu}t^{\lambda} = 0, \tag{IV.14}$$

i.e. whenever the motion of the test particle is confined to a characteristic surface. Since the Frobenius theorem ensures that the whole space is foliated into the characteristic surfaces, the above ambiguity for the test particle is removed whenever its initial velocity $\boldsymbol{t}$ is orthogonal to the characteristic vector $\hat{\boldsymbol{p}}$ at its initial position.

In other words: the autoparallel and geodesic curves coincide on the characteristic surfaces! Observe here that the parallel transport with respect to the standard connection $\boldsymbol{\Gamma}$ does not lead off the characteristic surfaces (cf. (II.28)).

As a result we see that the effect of the integrability condition consists in the partial elimination of the ambiguity of particle motion for the Riemann-Cartan structure. This partial goal shall now be converted into a total one.

2. *Transition to a Riemannian Structure*

The Frobenius condition (IV.4) acts even more effectively than could be supposed through the preceding results. Indeed, it enables us to get rid of the torsion completely such that the geometry becomes strictly Riemannian. But, surprisingly enough, the new Riemannian structure does not refer to the Riemannian part $\tilde{\boldsymbol{\Gamma}}$ of the standard connection $\boldsymbol{\Gamma}$ (II.52), which is the Levi-Civita connection of the standard metric $\boldsymbol{G}$ (II.42) used so far.

In order to find the new connection $\overset{*}{\boldsymbol{\Gamma}}$ *(characteristic connection),* reconsider first the standard connection $\boldsymbol{\Gamma}$ (II.52) for the case when the integrability condition holds

$$\Gamma^{\lambda}{}_{\mu\nu} = \tilde{\Gamma}^{\lambda}{}_{\mu\nu} + \hat{p}_\nu(\hat{p}_\mu f^\lambda - \hat{p}^\lambda f_\mu). \tag{IV.15}$$

The integrability condition also implies that there exists a function ϕ which may be used to form a gradient vector $\boldsymbol{C}$ in the following way

$$C_\mu = f_\mu + \phi\,\hat{p}_\mu, \quad \text{(a)} \qquad \partial_{[\mu} C_{\nu]} = 0. \quad \text{(b)} \tag{IV.16}$$

In terms of this gradient field, the standard connection Γ (IV.15) may be rewritten as

$$\Gamma^{\lambda}{}_{\mu\nu} = \tilde{\Gamma}^{\lambda}{}_{\mu\nu} + \hat{p}_\nu(C^\lambda \hat{p}_\mu - C_\mu \hat{p}^\lambda). \tag{IV.17}$$

The desired characteristic connection $\overset{*}{\boldsymbol{\Gamma}}$ emerges now through the following splitting of $\boldsymbol{\Gamma}$:

$$\Gamma^{\lambda}{}_{\mu\nu} = \overset{*}{\Gamma}{}^{\lambda}{}_{\mu\nu} + \overset{*}{K}{}^{\lambda}{}_{\mu\nu} \tag{IV.18}$$

where

$$\overset{*}{\Gamma}{}^{\lambda}{}_{\mu\nu} = \tilde{\Gamma}^{\lambda}{}_{\mu\nu} + C^\lambda \hat{p}_\mu \hat{p}_\nu - \hat{p}^\lambda(\hat{p}_\nu C_\mu + \hat{p}_\mu C_\nu) \quad \text{(a)} \qquad \overset{*}{K}{}^{\lambda}{}_{\mu\nu} = \hat{p}^\lambda \hat{p}_\mu C_\nu. \quad \text{(b)} \tag{IV.19}$$

Before one is allowed to deal with $\overset{*}{\boldsymbol{\Gamma}}$ in place of $\boldsymbol{\Gamma}$ as the bundle connection of τ_4, one must check whether the relevant properties of $\boldsymbol{\Gamma}$, constructed so carefully in Sects. II and III, are not destroyed again by the new $\overset{*}{\boldsymbol{\Gamma}}$. First, let us point out that the characteristic connection $\overset{*}{\boldsymbol{\Gamma}}$ has the same essential part γ (II.10) as the standard connection $\boldsymbol{\Gamma}$. Next, we want to look for the change of curvature. The splitting (IV.18) of $\boldsymbol{\Gamma}$ induces the following splitting of the curvature $\boldsymbol{R}$ (II.29)

$$R^{\lambda}{}_{\nu\mu\sigma} = \overset{*}{R}{}^{\lambda}{}_{\nu\mu\sigma} + \nabla_\mu \overset{*}{K}{}^{\lambda}{}_{\nu\sigma} - \nabla_\sigma \overset{*}{K}{}^{\lambda}{}_{\nu\mu} + \overset{*}{K}{}^{\lambda}{}_{\varrho\sigma} \overset{*}{K}{}^{\varrho}{}_{\nu\mu} - \overset{*}{K}{}^{\lambda}{}_{\varrho\mu} \overset{*}{K}{}^{\varrho}{}_{\nu\sigma} + 2 Z^{\varrho}{}_{\sigma\mu} \overset{*}{K}{}^{\lambda}{}_{\nu\varrho}. \tag{IV.20}$$

However, the particular structure of the contorsion $\overset{*}{\boldsymbol{K}}$ (IV.19 b) implies the vanishing of the quadratic terms on the right of (IV.20). Further, the covariant constancy of $\hat{\boldsymbol{p}}$ (II.38) together with the action of the torsion operator $\boldsymbol{Z}$ (II.6) yields

$$\nabla_\mu \overset{*}{K}{}^{\lambda}{}_{\nu\sigma} - \nabla_\sigma \overset{*}{K}{}^{\lambda}{}_{\nu\mu} = 2 Z^{\varrho}{}_{\mu\sigma} \overset{*}{K}{}^{\lambda}{}_{\nu\varrho}. \tag{IV.21}$$

Thus, we find from (IV.20) that the curvature remains unchanged

$$R^{\lambda}{}_{\nu\mu\sigma} = \overset{*}{R}{}^{\lambda}{}_{\nu\mu\sigma}. \tag{IV.22}$$

After this satisfactory results has been established, we have to show further that a metric $\overset{*}{\boldsymbol{G}}$ exists, such that the new $\overset{*}{\boldsymbol{\Gamma}}$ is just the Levi-Civita connection of $\overset{*}{\boldsymbol{G}}$, i.e.

$$\overset{*}{\nabla}_\lambda \overset{*}{G}_{\mu\nu} = 0. \tag{IV.23}$$

Such a Riemannian metric *(characteristic metric)* is readily found by first rewriting the gradient $\boldsymbol{C}$ as

$$C_\mu = \psi^{-1}\,\partial_\mu \psi = \partial_\mu \ln \psi \tag{IV.24}$$

and then observing that the rescaled characteristic vector $\boldsymbol{p}$

$$p_\mu = \psi\,\hat{p}_\mu \tag{IV.25}$$

is now constant with respect to the characteristic connection $\overset{*}{\boldsymbol{\Gamma}}$

$$\overset{*}{\nabla}_\lambda p_\mu = 0. \tag{IV.26}$$

This implies that $\boldsymbol{p}$ is a gradient vector and ψ is an integrating factor for $\hat{\boldsymbol{p}}$. Further, following the idea of (II.43) we fix the new metric *(characteristic metric)* as

$$\overset{*}{G}_{\mu\nu} = c^2 B_{\mu\nu} - p_\mu p_\nu, \tag{IV.28}$$

which then clearly meets with the constancy requirement (IV.23). The characteristic metric defines now the Riemannian structure which we were after. Observe also that the rescaled characteristic vector is still of unit length with respect to the new metric $\overset{*}{\boldsymbol{G}}$:

$$\overset{*}{p}{}^\mu p_\mu = \overset{*}{G}_{\mu\nu} \overset{*}{p}{}^\mu \overset{*}{p}{}^\mu = -1, \quad \text{(a)} \qquad \overset{*}{p}{}^\mu = \psi^{-1}\,\hat{p}^\mu. \quad \text{(b)} \tag{IV.29}$$

Finally, one surely wonders whether the generalized conformality of the Riemann-Cartan structure is reverted again into the ordinary conformally flat structure of the Riemannian case. A short calculation shows that the Weylian (III.26) indeed vanishes for the present Riemannian case as well in the intrinsic ($n = 3$) as in the extrinsic ($n = 4$) version. Observe here that the curvature tensor $\boldsymbol{R}$, the Ricci $\mathcal{R}$, the scalar $\mathcal{S}$, and the fibre metric $\bar{G}$ are the same objects in both the Riemannian and non-Riemannian approaches, whereas the connections Γ, $\overset{*}{\Gamma}$ and the metric tensors $\boldsymbol{G}$, $\overset{*}{\boldsymbol{G}}$ are different along the characteristic directions. In any case, the Riemannian structure is a conformally flat one, albeit the Riemannian metric $\overset{*}{\boldsymbol{G}}$ will in general not be found proportional to the Euclidean $\boldsymbol{g}$. However, a general coordinate transformation must exist which makes $\overset{*}{\boldsymbol{G}}$ proportional to $\boldsymbol{g}$. In special cases (e.g. dimeron solution) this proportionality may be existent immediately for the original Cartesian coordinates! The characteristic surfaces now appear as 3-dimensional Riemannian submanifolds equipped with a geometry of constant curvature.

The present transition to a strictly Riemannian structure evidently cures also the pathology of the previous Riemann-Cartan geometry concerning the motion of test particles. Therefore, in view of some application of the present mathematical structure to a real physical situation, one surely will prefer the characteristic connection $\overset{*}{\boldsymbol{\Gamma}}$ over the standard connection $\boldsymbol{\Gamma}$ whenever this is allowed by the integrability condition. As for the new geodesic curves on the integral surfaces of $\bar{\Delta}$, one finds for the corresponding equation of motion

$$\frac{\overset{*}{\mathrm{D}}{}^2 x^\mu}{\mathrm{d}s^2} = \frac{\mathrm{D}^2 x^\mu}{\mathrm{d}s^2} - \overset{*}{K}{}^\mu{}_{\nu\lambda}\frac{\mathrm{d}x^\nu}{\mathrm{d}s}\frac{\mathrm{d}x^\lambda}{\mathrm{d}s} = 0\,. \qquad \text{(IV.30)}$$

Therefore, the geodesic curves of the characteristic metric $\overset{*}{\boldsymbol{G}}$ coincide with the previous curves (IV.12) on the characteristic surface and hence they may be considered as the proper continuation of the latter ones to the whole embedding space.

V. Yang-Mills Equations

Up to now we were mainly concerned with general gauge field *configurations* of the trivializable type. In the following, the interest shall be concentrated upon trivializable *solutions* of the free Yang-Mills equations. It turns out that these field equations represent a sufficient (but not necassary) condition for the existence of the characteristic surfaces. On the other hand, the results of the preceding sections imply the existence of a strictly Riemannian structure in addition to the more general and always existing Riemann-Cartan structure whenever the characteristic distribution is integrable. Thus the combination of these two arguments says that the validity of the Yang-Mills equations ensures the existence of a Riemannian structure. The general properties of such a "Riemann-Yang-Mills" structure shall be the aim of the subsequent investigations.

1. Geometric Meaning of the Field Equations

Assume that the trivializable gauge fields are constrained now to satisfy the free Yang-Mills equations

$$D^\mu F_{i\mu\nu} = 0\,. \qquad \text{(V.1)}$$

First we shall look for the geometric implication of these field equations.

For any trivializable gauge field $\boldsymbol{A}$ the intrinsic curvature $\boldsymbol{F}$ is determined by the extrinsic curvature $\boldsymbol{B}$ according to (II.3a). Therefore the Yang-Mills equations (V.1) may be rewritten in terms of the $\boldsymbol{B}$ fields as

$$D^\mu F_{i\mu\nu} \equiv \varepsilon_i{}^{jk} B_{k\mu}(D^\mu B_{j\nu} - \hat{P}^\mu{}_\nu(D^\lambda B_{j\lambda})) = 0\,. \qquad \text{(V.2)}$$

Introducing here an intermediate object $\boldsymbol{T}$ through

$$D_\mu B_{j\nu} - \hat{P}_{\mu\nu}(D^\lambda B_{j\lambda}) = T^\lambda{}_{\nu\mu} B_{j\lambda}\,, \qquad \text{(V.3)}$$

this object is then readily found to be related to the essentiall part γ (II.10) through

$$T^\lambda{}_{\nu\mu} = \gamma^\lambda{}_{\nu\mu} - \hat{P}_{\mu\nu}\gamma^{\lambda\sigma}{}_\sigma\,. \qquad \text{(V.4)}$$

By the use of the essential part γ in place of the proper connection $\boldsymbol{\Gamma}$, we have avoided here the emergence of superfluous components of $\boldsymbol{T}$ such that

$$\hat{p}_\lambda T^\lambda{}_{\nu\mu} = 0\,. \qquad \text{(V.5)}$$

This forces $\boldsymbol{T}$ to be symmetric

$$T^\lambda{}_{\mu\nu} = T^\lambda{}_{\nu\mu}\,. \qquad \text{(V.6)}$$

With these arrangements, the Yang-Mills equations (V.1) now read

$$D^\mu F_{i\mu\nu} \equiv -F_i^{\lambda\mu} T_{\lambda\mu\nu} = 0\,. \qquad \text{(V.7)}$$

In this form, the Yang-Mills problem immediately suggests its general solution: Observe that, for any non-degenerate point $\mathbf{x}$ of base space, the curvature coefficients F_i are three linearly independent 2-forms on $\bar{\Delta}_x$. Therefore, the most general solution for $\boldsymbol{T}$ is

$$T_{\lambda\mu\nu} = S_{\lambda\mu\nu} + S_{\lambda\mu}\hat{p}_\nu + S_{\lambda\nu}\hat{p}_\mu + S_\lambda \hat{p}_\mu \hat{p}_\nu . \quad \text{(V.8)}$$

The $\boldsymbol{S}$ objects emerging here are living completely in the corresponding tensor bundles associated to $\bar{\tau}_4$, e.g.

$$\hat{p}^\lambda S_{\lambda\mu\nu} = \hat{p}^\lambda S_{\lambda\mu} = \hat{p}^\lambda S_\lambda = 0 . \quad \text{(V.9)}$$

They must be totally symmetric in order that the Yang-Mills equations are satisfied, i.e.

$$S_{\lambda\mu\nu} = S_{\mu\lambda\nu} = S_{\mu\nu\lambda}, \quad \text{(a)}$$
$$S_{\mu\lambda} = S_{\lambda\mu} . \quad \text{(b)} \qquad \text{(V.10)}$$

Once the general form of the $\boldsymbol{T}$ object is known, the essential part γ is readily found from (V.4) as

$$\gamma^\lambda{}_{\nu\mu} = S^\lambda{}_{\nu\mu} + S^\lambda{}_\mu \hat{p}_\nu + S^\lambda{}_\nu \hat{p}_\mu + S^\lambda \hat{p}_\mu \hat{p}_\nu - \tfrac{1}{2}\hat{P}_{\nu\mu}(S^{\lambda\sigma}{}_\sigma - S^\lambda) . \quad \text{(V.11)}$$

Consequently, the standard connection $\boldsymbol{\Gamma}$ becomes (cf. (II.36)):

$$\Gamma^\lambda{}_{\nu\mu} = \gamma^\lambda{}_{\nu\mu} - \hat{p}^\lambda (S_{\nu\mu} + S_\nu \hat{p}_\mu) . \quad \text{(V.12)}$$

At this stage, the geometric meaning of the Yang-Mills equations becomes obvious now: Consider the derivative of the characteristic vector field which is found from (V.12) as

$$\partial_\mu \hat{p}_\nu = \Gamma^\lambda{}_{\nu\mu}\hat{p}_\lambda = S_{\nu\mu} + S_\nu \hat{p}_\mu . \quad \text{(V.13)}$$

This implies that the characteristic distribution $\bar{\Delta}$ becomes integrable (cf. (IV.4)) as a consequence of the validity of the field equations:

$$\partial_{[\mu}\hat{p}_{\nu]} = \hat{p}_{[\mu} S_{\nu]} . \quad \text{(V.14)}$$

The shape of the corresponding integral surfaces is determined by the objects $S_{\mu\nu}$ through

$$\hat{P}^\lambda{}_\mu \partial_\lambda \hat{p}_\nu = S_{\mu\nu} . \quad \text{(V.15)}$$

Observe that the *torsion vector* $\boldsymbol{S}$ defined through

$$S_\nu = -\hat{p}^\mu \partial_\mu \hat{p}_\nu \quad \text{(V.16)}$$

is different from zero only when the characteristic lines are not straight (with respect to the canonical connection $\overset{\circ}{\omega}$ over $\boldsymbol{E}_4$). Therefore, the standard connection $\boldsymbol{\Gamma}$ exhibits a non-vanishing torsion $\boldsymbol{Z}$

$$Z^\lambda{}_{\mu\nu} = \hat{p}^\lambda \hat{p}_{[\mu} S_{\nu]} \quad \text{(V.17)}$$

which measures the deviation of the characteristic lines from the Euclidean geodesics. (Remember here that the characteristic curves are geodesics of the characteristic metric $\overset{*}{\boldsymbol{G}}$ in any case!)

A further consequence of (V.13) refers to the difference of two trivializable gauge fields $\boldsymbol{A}$, which are not gauge copies of one another. If they are sharing the same characteristic surfaces, then their standard connections $\boldsymbol{\Gamma}$ are based upon the same objects $S_{\mu\nu}$ and S_μ but they must differ with respect to the totally symmetric object $S_{\mu\nu\lambda}$!

2. *Riemann-Yang-Mills Structure*

The preceding arguments show that the Yang-Mills equations essentially represent a sufficient integrability condition. Therefore, they simultaneously ensure the existence of a strictly Riemannian structure. From the physical point of view, it seems highly welcome to be able to express the rather formal integrability condition by a dynamical field equation. The result of this is a sort of a Riemann-Yang-Mills space structure, which is built upon a conformally flat background geometry.

The transition to the Riemannian connection $\overset{*}{\boldsymbol{\Gamma}}$ is well prepared by the arguments of Section IV.2. Using here the standard connection $\boldsymbol{\Gamma}$ (V.12), which has been deduced from the validity of the free Yang-Mills equations, we simply arrive at its characteristic counterpart $\overset{*}{\boldsymbol{\Gamma}}$ via (IV.18, 19)

$$\overset{*}{\Gamma}{}^\lambda{}_{\mu\nu} = \Gamma^\lambda{}_{\mu\nu} - \overset{*}{p}{}^\lambda p_\mu C_\nu . \quad \text{(V.18)}$$

However, in order to avoid some lengthy expresssions, we do not naively introduce here the result (V.12) for the standard connection $\boldsymbol{\Gamma}$; rather, we prefer to express $\overset{*}{\boldsymbol{\Gamma}}$ in terms of that conformally flat connection $\bar{\boldsymbol{\Gamma}}$ which is defined through

$$\bar{\Gamma}^\lambda{}_{\mu\nu} = g^\lambda{}_\mu C_\nu + g^\lambda{}_\nu C_\mu - C^\lambda g_{\mu\nu} . \quad \text{(V.19)}$$

Clearly, this is the Levi-Civita connection of the conformally flat metric $\bar{\boldsymbol{G}}$

$$\bar{G}_{\mu\nu} = \psi^2 g_{\mu\nu} . \quad \text{(V.20)}$$

The conformally flat $\bar{\boldsymbol{\Gamma}}$ enters the characteristic connection $\overset{*}{\Gamma}$ (V.18) quite simply by evoking the relationship between the gradient $\boldsymbol{C}$ and the torsion vector $\boldsymbol{S}$ (cf. (IV.16) and (V.14))

$$S_\mu = \hat{P}^\nu{}_\mu C_\nu , \quad \text{(a)}$$
$$\Phi = -(\hat{p}^\nu C_\nu) . \quad \text{(b)} \qquad \text{(V.21)}$$

Introducing further the deviations $(s_{\lambda\mu\nu}, s_{\lambda\mu})$ of the symmetric tensors $(S_{\lambda\mu\nu}, S_{\lambda\mu})$ from their conformally flat values $(\bar{S}_{\lambda\mu\nu}, \bar{S}_{\lambda\mu})$ given by

$$\begin{aligned} \bar{S}_{\lambda\mu\nu} &= \hat{P}_{\lambda\mu} S_\nu + \hat{P}_{\mu\nu} S_\lambda + \hat{P}_{\nu\lambda} S_\mu\,, &\text{(a)}\\ \bar{S}_{\mu\nu} &= \Phi \hat{P}_{\mu\nu}\,, &\text{(b)} \end{aligned} \tag{V.22}$$

one puts

$$\begin{aligned} S_{\lambda\mu\nu} &= \bar{S}_{\lambda\mu\nu} + s_{\lambda\mu\nu}\,, &\text{(a)}\\ S_{\lambda\nu} &= \bar{S}_{\lambda\nu} + s_{\lambda\nu}\,, &\text{(b)} \end{aligned} \tag{V.23}$$

and then one finds for the characteristic connection

$$\overset{*}{\Gamma}{}^{\lambda}{}_{\mu\nu} = \bar{\Gamma}^{\lambda}{}_{\mu\nu} + \bar{Q}^{\lambda}{}_{\mu\nu}\,. \tag{V.24}$$

Here, the deviation $\bar{\boldsymbol{Q}}$ of $\overset{*}{\boldsymbol{\Gamma}}$ from $\bar{\boldsymbol{\Gamma}}$ is composed of the deviations $\boldsymbol{s}$ from $\bar{\boldsymbol{S}}$ according to

$$\bar{Q}^{\lambda}{}_{\mu\nu} = s^{\lambda}{}_{\mu\nu} + s^{\lambda}{}_{\mu}\,\hat{p}_\nu + s^{\lambda}{}_{\nu}\,\hat{p}_\mu - \hat{p}^{\lambda}\,s_{\mu\nu} - \tfrac{1}{2}\,\hat{P}_{\mu\nu}\,s^{\lambda\sigma}{}_{\sigma}\,. \tag{V.25}$$

The partitioning of $\overset{*}{\boldsymbol{\Gamma}}$ into the conformally flat contribution $\bar{\boldsymbol{\Gamma}}$ and the remainder $\bar{\boldsymbol{Q}}$ reflects the circumstance that there is no torsion $\boldsymbol{Z}$ in the Riemannian approach whereas we nevertheless must have a non vanishing torsion vector $\boldsymbol{S}$ in the theory, because the characteristic lines are expected to be curved in the quite general case (cf. (V.16))! It is just the splitting (V.24) of $\overset{*}{\boldsymbol{\Gamma}}$, which elegantly reconciles this apparent contradiction by absorbing the torsion vector into the conformally flat part $\bar{\boldsymbol{\Gamma}}$. The remainder $\bar{\boldsymbol{Q}}$ (V.25) then becomes completely free of torsion objects but otherwise looks like the standard connection $\boldsymbol{\Gamma}$ (V.12).

Obviously, the object $\bar{Q}$ defines some additional structure living on the conformally flat background geometry. The corresponding background metric is given by (V.20) and satisfies

$$\overset{*}{\nabla}_\lambda\, \bar{G}_{\mu\nu} = 2\,Q_{\mu\nu\lambda}\,. \tag{V.26}$$

Resolving this equation for the characteristic connection yields the splitting (V.24) with $\bar{\boldsymbol{Q}}$ being given in terms of $\boldsymbol{Q}$ as [11]

$$\bar{Q}^{\lambda}{}_{\mu\nu} = Q_{\mu\nu}{}^{\bar{\lambda}} - (Q^{\bar{\lambda}}{}_{\nu\mu} + Q^{\bar{\lambda}}{}_{\mu\nu})\,. \tag{V.27}$$

Thus, $\boldsymbol{Q}$ is revealed as the symmetric part Σ' of $\bar{\boldsymbol{Q}}$; i.e.

$$\begin{aligned} Q_{\mu\nu\lambda} &= -\Sigma'_{\bar{\mu}\nu\lambda} \equiv -\tfrac{1}{2}(\bar{Q}_{\bar{\mu}\nu\lambda} + \bar{Q}_{\bar{\nu}\mu\lambda})\,, &\text{(a)}\\ \Sigma'^{\lambda}{}_{\mu\nu} &= s^{\lambda}{}_{\mu\nu} + s^{\lambda}{}_{\mu}\,\hat{p}_\nu - \tfrac{1}{4}(\hat{P}_{\mu\nu}\,s^{\lambda\sigma}{}_{\sigma} + \hat{P}^{\lambda}{}_{\nu}\,s_{\mu\sigma}{}^{\sigma})\,. &\text{(b)} \end{aligned} \tag{V.28}$$

Evidently, the most simple solutions are represented by the *conformally flat type*. By its very definition this class of solutions consists of pure background configurations, i.e. $\bar{\boldsymbol{Q}} \equiv 0$ and hance $\overset{*}{\boldsymbol{\Gamma}} \equiv \bar{\boldsymbol{\Gamma}}$. The most general, non-trivial solution of this type has already been identified and investigated in the previous papers [1, 2]. It is the well-known di-meron configuration, the geometric and topological properties of which are well understood now. Therefore, a very brief discussion, based upon two new aspects, may be sufficient here.

First, observe that the second rank object $S_{\mu\nu}$ (V.22 b) becomes proportional to the projector $\hat{\boldsymbol{P}}$, which implies that the characteristic surfaces are 3-spheres of radius Φ^{-1} (Figure 1). Since the function $\Phi(x)$ is constant over anyone of the 3-spheres, its gradient must determine the characteristic vector $\hat{\boldsymbol{p}}$. Indeed, a simple calculation yields

$$\partial_\mu \Phi = -\tfrac{1}{2}(S^\lambda S_\lambda - \Phi^2 - \psi^2/c^2)\,\hat{p}_\mu\,, \tag{V.29}$$

where the torsion vector S_λ and the inverse radius Φ are related to the conformal gradient $\boldsymbol{C}$ (IV.24) through (V.21).

Second, observe that the fibre metric $\bar{\boldsymbol{G}}$ (II.45) becomes isotropic for the pure background solutions

$$G_{\mu\nu} = \overset{*}{G}_{\mu\nu} + p_\mu p_\nu = \psi^2 \hat{P}_{\mu\nu}\,. \tag{V.30}$$

Conversely, one readily recognizes by this argument that any isotropic solution necessarily must be conformally flat and hence must coincide with a di-meron configuration. Thus we arrive at the conclusion that the non-trivial background geometry always is conformally flat and isotropic. Moreover, it is described by a symmetric Riemannian structure. In this sense, the background geometry exhibits the maximal symmetry which can be found in the whole set of solutions.

Appendix

Comparison of Geodesic and Autoparallel Curves

In a non-Riemannian structure the autoparallel and geodesic curves may differ considerably. This is also true for the present almost Riemannian structure, which becomes strictly Riemannian when it is restricted to the characteristic surfaces. As an example, we study the two types of curves for the dimeron configuration.

First, consider the geodesic equation (IV.30) for the Riemannian case

$$\frac{d^2 x^\mu}{ds^2} + \bar{\Gamma}^{\mu}{}_{\nu\lambda}\,\frac{dx^\nu}{ds}\,\frac{dx^\lambda}{ds} = 0\,, \tag{A.1}$$

where the conformally flat connection $\bar{\boldsymbol{\Gamma}}$ is given by (V.19) and the conformal scale factor ψ has been found as [1]

$$\psi = \frac{|\boldsymbol{a}-\boldsymbol{b}|^2}{|\boldsymbol{x}-\boldsymbol{a}|\,|\boldsymbol{x}-\boldsymbol{b}|}\,. \tag{A.2}$$

The line element, used in (A.1), is determined by the conformally flat metric $\bar{\boldsymbol{G}}$ (V.20) through

$$ds^2 = -\,\bar{G}_{\mu\nu}\,dx^\mu\,dx^\nu\,. \tag{A.3}$$

Since we effectively are working in a Euclidean space, it is more convenient to use the Euclidean line element

$$d\sigma^2 = -\,g_{\mu\nu}\,dx^\mu\,dx^\nu, \tag{A.4}$$

which casts the equation of motion (A.1) into the following form

$$\frac{d^2x^\mu}{d\sigma^2} + \hat{h}^\mu{}_\varrho\,\bar{\Gamma}^\varrho{}_{\nu\lambda}\,\frac{dx^\nu}{d\sigma}\,\frac{dx^\lambda}{d\sigma} = 0\,. \tag{A.5}$$

The projector $\hat{\boldsymbol{h}}$ occurring here annihilates the "four velocity" $dx^\lambda/d\sigma$, i.e.

$$\hat{h}^\mu{}_\varrho = g^\mu{}_\varrho + \frac{dx^\mu}{d\sigma}\,\frac{dx_\varrho}{d\sigma}\,. \tag{A.6}$$

Introducing now the conformally flat $\bar{\boldsymbol{\Gamma}}$ into (A.5) yields an equation of motion which can be formally put into the usual shape encountered in special relativity:

$$\frac{d^2x^\mu}{d\sigma^2} = F^\mu\,. \tag{A.7}$$

Here the "four-force" $\boldsymbol{F}$ is orthogonal to the four-velocity, i.e.

$$F^\mu = -\,\hat{h}^\mu{}_\varrho\,C^\varrho, \tag{A.8}$$

as it is necessary for "relativistic" consistency.

However, for the numerical computations it is advantageous to deal with an equation of motion of the Newtonian type. The latter one is obtained from the relativistic form (A.7) with the aid of a parameter transformation which relates the "proper time" (σ) to the "Newtonian time" (τ) in the following way

$$d\sigma/d\tau = \psi\,. \tag{A.9}$$

Thus, the Newtonian equation of motion is found as

$$d^2x^\mu/d\tau^2 = -\,\partial_\mu\,\mathcal{V}, \tag{A.10}$$

where the potential $\mathcal{V}$ is given by

$$\mathcal{V} = -\tfrac{1}{2}\,\psi^2, \tag{A.11}$$

and obviously is due to the action of two attractive centers located at the meron positions $\boldsymbol{a}$, $\boldsymbol{b}$ (Figure 2).

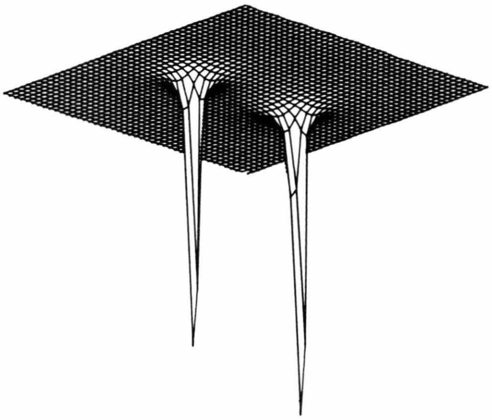

Fig. 2. The potential $\mathcal{V}$ (A.11) is due to two attractive centers located at the meron positions $\boldsymbol{a}$, $\boldsymbol{b}$. The solutions of the geodesic equation (A.1) are represented by the Newtonian particle trajectories for this potential. Since the total energy vanishes (A.13), the test particle is bound by the two merons.

The "non-relativistic" eqution of motion (A.10) and its original "relativistic" counterpart (IV.11) provides us now with a sufficiently qualitative picture of the geodesic lines: contraction of (A.10) by the velocity $dx_\mu/d\tau$ gives the energy conservation law in the form

$$\frac{d\mathcal{E}}{d\tau} \equiv \frac{d}{d\tau}(\mathcal{T}+\mathcal{V}) = 0\,. \tag{A.12}$$

Further, the total energy $\mathcal{E} = \mathcal{T} + \mathcal{V}$ is found to be zero, because the "speed" of the test particle equals the conformal scale function ψ (cf. (A.9)) which gives for the "kinetic energy" $\mathcal{T}$

$$\mathcal{T} = -\frac{1}{2}\,\frac{dx^\mu}{d\tau}\,\frac{dx_\mu}{d\tau} = +\tfrac{1}{2}\,\psi^2 = -\,\mathcal{V}\,. \tag{A.13}$$

The conclusion from this result is that the test particle, following a geodesic line, is bound by the two merons and hence will not be able to escape to infinity.

This conclusion may be made more precise by reconsidering the original "relativistic" form of the equation of motion (IV.11). Since both the unit vectors $\boldsymbol{t}$ and $\boldsymbol{p}$ are parallel transported along a geodesic curve, the (Euclidean) angle α enclosed by them is constant along the geodesic. More concretely, the angle α by which the sequence of characteristic surfaces is cut through by a given geodesic is constant along that curve. But on the other hand, any characteristic surface encloses just one of the two meron centers (Figure 1). Therefore, any geodesic curve must necessarily terminate on a meron center or is a closed curve contained in a characteristic surface. The general-

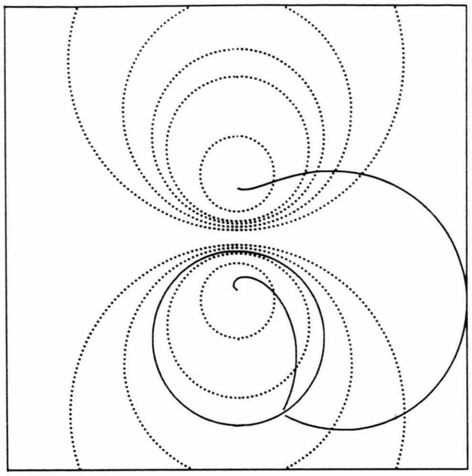

Fig. 3. The 2-dimensional Newtonian problem (A.10, 11) admits circular trajectories for the particle motion (dotted lines). The circles represent the characteristic surfaces from the geometric point of view. An arbitrary trajectory (solid line) cuts the circles by a constant angle α.

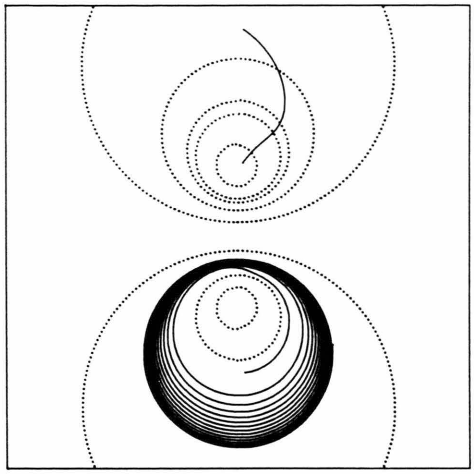

Fig. 4. The autoparallel trajectories tend to cut the characteristic surface more and more orthogonally if they approach an attractive center (upper part). For the autoparallel motion off the attractive center, the test particle has the tendency to reach a circular path on a characteristic surface (lower part).

ization of this result to an arbitrary multi-meron solution (if existing) is evident.

Figure 3 shows the numerical verification based upon the Newtonian equation (A.10) for two dimensions. Such a regular pattern of the particle motion could surely not be expected by considering the classical Newtonian problem (A.10, 11) alone; the crucial point here is the complementation of the purely Newtonian approach by the geodesic problem for the dimeron background.

Next, consider the autoparallel curves (IV.12a) for the Riemann-Cartan geometry of the dimeron configuration. The equation of motion for the autoparallels differs from that for the geodesics by the contorsion $\overset{*}{\boldsymbol{K}}$ (IV.19b). Consequently this term produces some modification of the Newtonian form (A.10, 11) of the geodesic equation. After some simple calculations, the autoparallel counterpart of the geodesic equation is found as

$$\frac{d^2x^\mu}{d\tau^2} = -\partial_\mu \mathcal{V} - (\hat{h}^\mu{}_\varrho\, p^\varrho)\left(p_\nu \frac{dx^\nu}{d\tau}\right)\left(C_\lambda \frac{dx^\lambda}{d\tau}\right). \quad \text{(A.14)}$$

As a short check, one easily recognizes that the additional term on the right of (A.14) vanishes (i) for a characteristic line and (ii) for the motion on the characteristic surfaces. Further, the addtional term does not invalidate the energy conservation (A.12, 13) on behalf of the projector property of $\hat{\boldsymbol{h}}$ (A.6).

A qualitative picture of the autoparallels as solutions to (A.14) comes again from the discussion of the angle α enclosed by the tangent and characteristic vectors $\boldsymbol{t}, \boldsymbol{p}$. Since α is constant along the autoparallels, we readily obtain a constant of the motion through

$$\cos\alpha = G_{\mu\nu}\,\hat{p}^\mu\, t^\nu . \quad \text{(A.15)}$$

It is useful to express the Riemann-Cartan angle α (A.15) through its Euclidean analogue α_E

$$\cos\alpha_E = g_{\mu\nu}\,\hat{p}^\mu\,\hat{t}^\nu, \quad (\hat{t}^\mu = t^\mu\, ds/d\sigma)\,, \quad \text{(A.16)}$$

which yields

$$\cos^2\alpha_E = \psi^2\cos^2\alpha\,(\sin^2\alpha + \psi^2\cos^2\alpha)^{-1}. \quad \text{(A.17)}$$

Clearly, both angles α, α_E are identical for $\alpha = 0, \frac{\pi}{2}$. But now release the "test particle" by some angle $0 < \alpha < \frac{\pi}{2}$ such that the potential $\mathcal{V}$ is decreasing, i.e. the test particle is pushed to approach a meron center. In this case, α_E tends to zero which means that an autoparallel curve will approach the meron center more quickly and directly than a geodesic line. On the other hand, if the test particle is pushed off a meron center, such that the potential $\mathcal{V}$ is increasing, the Euclidean angle α_E tends to become a right angle, i.e. the test particle is reluctant to approach the other meron center (as in the geodesic case) but prefers to move along a stationary circle around the first meron center (Figure 4).

[1] R. Brucker and M. Sorg, Z. Naturforsch. **41a,** 571 (1986).
[2] R. Brucker and M. Sorg, Z. Naturforsch. **42a,** 521 (1987).
[3] F. W. Hehl et al., Rev. Mod. Phys. **48,** 393 (1976).
[4] The index bracket [...] means the complete antisymmetrization of the enclosed indices.
[5] S. Kobayashi and K. Nomizu, Foundations of Differential Geometry, Vol. 1 and 2. Interscience Publishers, 1969.
[6] Since we want to have the metric dimensionless, we include some length parameter c on the right of (II.45).
[7] Y. Choquet-Bruhat et. al., Analysis, Manifolds and Physics, North-Holland, 1982.
[8] R. Sexl and H. K. Urbantke, Gravitation und Kosmologie, BI-Wissenschaftsverlag, 1983.
[9] B. Chen, Geometry of Submanifolds, Marcel Dekker, Inc., New York 1973.
[10] C. von Westenholz, Differential Forms in Mathematical Physics, North-Holland 1978.
[11] Indices with a bar are raised or lowered by use of the background metric $\bar{\boldsymbol{G}}$ (V.20).